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Why can the weighting parameter of the Muskingum channel routing method be negative?

J. Szilagyi

Research Centre for Water Resources Development, VITUKI National Hydrological Forecasting Service, H-1453 Budapest, P.O. Box 27, Hungary

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ABSTRACT

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For a typical river reach, observations often show that the peak value of the outflow is lower than the synchronous value of the inflow. This is caused by the adoption of simplified rating curves or/and the discrete nature of water stage measurements. In this situation, the most accurate estimation of storage in the reach, using the conventional Muskingum method, is when the weighting parameter θ is negative.

INTRODUCTION

The conventional Muskingum channel routing method estimates the storage in a river reach at any given time as follows (Nash, 1959; Dooge et al., 1982):

$$S = K[\theta \cdot I + (1 - \theta) \cdot Q] \tag{1}$$

The rate of change of storage within the reach at any instant is given by

$$\mathrm{d}S/\mathrm{d}t = I - Q \tag{2}$$

Substituting eqn. (1) into eqn. (2), and integrating results in the solution

$$Q(t) = \frac{Q_0}{1-\theta}\exp\left[-\frac{t}{K(1-\theta)}\right] + \frac{1}{1-\theta}\frac{1}{K(1-\theta)}\int_0^t I(\tau) \times \exp\left[-\frac{t-\tau}{K(1-\theta)}\right]\mathrm{d}\tau - \frac{\theta}{1-\theta}I(t) \tag{3}$$

Correspondence to: J. Szilagyi, Research Centre for Water Resources Development, VITUKI National Hydrological Forecasting Service, H-1453 Budapest, P.O. Box 27, Hungary.

NOTATION

C_i	the ith coefficient in a finite difference scheme
I	inflow
I_r	reduced inflow
K	storage coefficient equal to the propagation time of discharges
Q	outflow
Q_r	reduced outflow
Q_0	initial steady flow rate
r	dimensionless time interval ratio
S	storage volume
t	time
Δt	time interval of discretization
θ	dimensionless weighting parameter
τ	dummy variable of integration
$\|\|$	absolute value of quantity between lines

where $Q_0 = I(0) = Q(0)$ is the initial condition (Chang et al., 1983). Without loss of generality, reduced discharges are introduced by subtracting the initial steady flow, Q_0, from the inflow and outflow values. The solution of the reduced system with the initial condition $I_r(0) = Q_r(0) = 0$ becomes

$$Q_r(t) = \frac{1}{1-\theta}\frac{1}{K(1-\theta)}\int_0^t I_r(\tau)\exp\left[-\frac{t-\tau}{K(1-\theta)}\right]\mathrm{d}\tau - \frac{\theta}{1-\theta}I_r(t) \quad (4)$$

Provided $0 < \theta \leqslant 1$, the last term of the solution reduces the outflow, thus indicating the possibility of $Q(t) < Q_0$, or $Q_r(t) < 0$. In the case $\theta \leqslant 0$, such a negative dip in the outflow does not occur (Chang et al., 1983).

THE EFFECT OF THE MEASUREMENT PROCEDURE ON THE VALUE OF θ

A general case is shown in Fig. 1, where the storages in the reach are seen to increase up to time t_2. If $\theta = 1$, then eqn. (1) reduces to $S = K \cdot I$, i.e. the peak of the storage is at t_1. If $\theta = 0$, then eqn. (1) yields $S = K \cdot Q$ i.e. the peak of the storage is at t_3. If the appropriate value for θ is chosen, where $0 < \theta < 1$, then the occurrence of the maximum value of the storage S at the correct point in time, i.e. at t_2, can be achieved.

An alternative situation depicted in Fig. 2, where the storage peak time t_2 occurs after the outflow peak time t_3, can often be observed in practice.

For operational river forecasting, stages are recorded and subsequently the corresponding discharges are deduced with the help of a simplified rating curve which does not contain a loop. Thus during the time interval t_3–t_2, of Fig. 2, the stages are falling at both the inflow and outflow sections. As a result

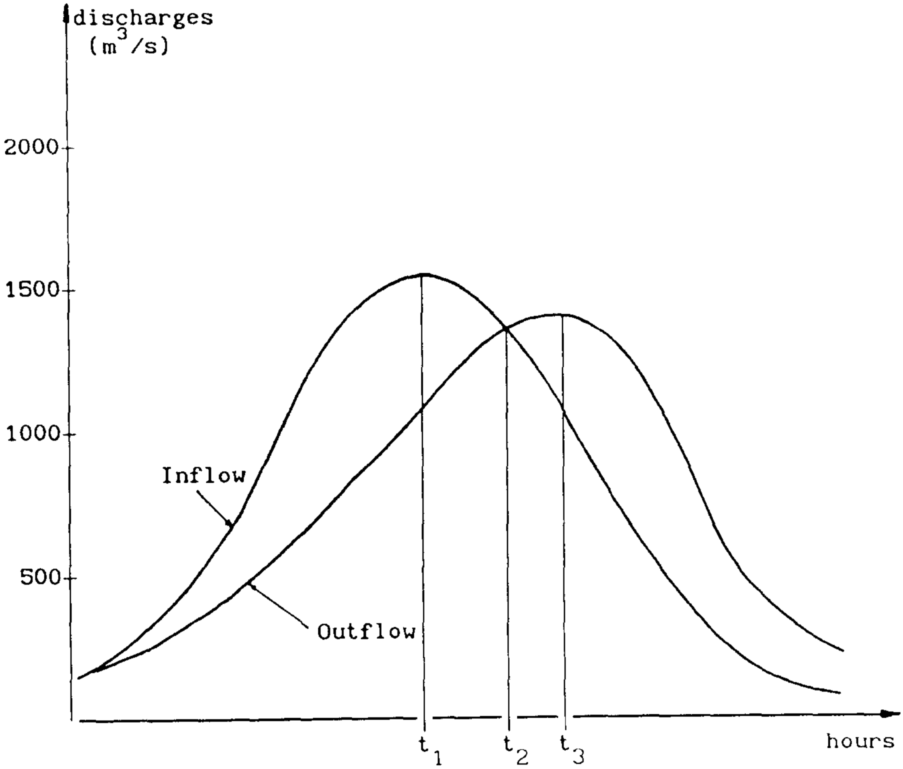

Fig. 1. Typical flood hydrographs for a river reach, with $t_1 < t_2 < t_3$.

of the simplification of the rating curve, the corresponding discharges are also decreasing in this time interval. However, over the same time interval, the stored water volume S is increasing until time t_2. Provided the river reach in question is short, such that the water surface profile between the upstream and downstream sections of the reach at any given time can be approximated by a straight line, then a physical contradiction arises. Namely, it is impossible that the storage S increases monotonically over the time interval t_3–t_2 while simultaneously the water surface level drops monotonically over the whole length of the reach.

However, if an accurate rating curve which contained a loop were applied instead, then the peak time t_3 of the outflow would occur after the peak time t_2 of the storage S, as shown in Fig. 1.

The situation can also arise where an appropriate (more realistic) rating curve is available but where the stages are measured at discrete time steps, which gives rise to errors in the interpolation procedure. In reality, therefore,

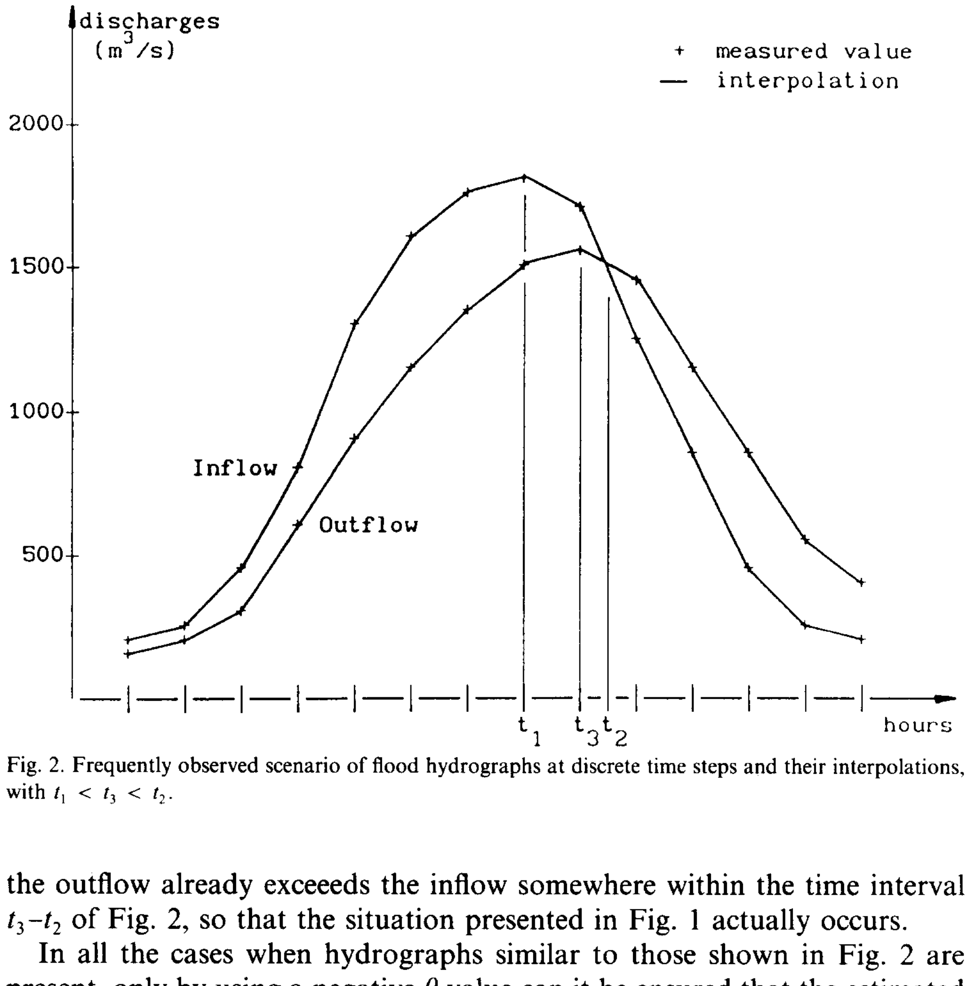

Fig. 2. Frequently observed scenario of flood hydrographs at discrete time steps and their interpolations, with $t_1 < t_3 < t_2$.

the outflow already exceeeds the inflow somewhere within the time interval t_3–t_2 of Fig. 2, so that the situation presented in Fig. 1 actually occurs.

In all the cases when hydrographs similar to those shown in Fig. 2 are present, only by using a negative θ value can it be ensured that the estimated storage peak follows the inflow and outflow maxima in time.

This can be proved as follows. Let us suppose $\theta < 0$, so that

$$S = K[\theta \cdot I + (1 - \theta) \cdot Q] = K[|\theta| \cdot (Q - I) + Q] \quad (5)$$

From Fig. 2 it can be seen that until time t_2, $Q < I$, so that the term $|\theta| \cdot (Q - I)$ is negative. The maximum value of the stored water volume S can occur after the time t_3 of the peak outflow provided the value of the term $||\theta| \cdot (Q - I)|$ is decreasing more rapidly in time than the value of Q. This condition is ensured by using an appropriate negative value of θ.

A practical example of the application of such an appropriate negative value of θ (where $\theta = -1.83$, obtained by the least squares method), is shown in Fig. 3, where eqn. (1) is used to obtain the 'estimated' storage S_e, and a

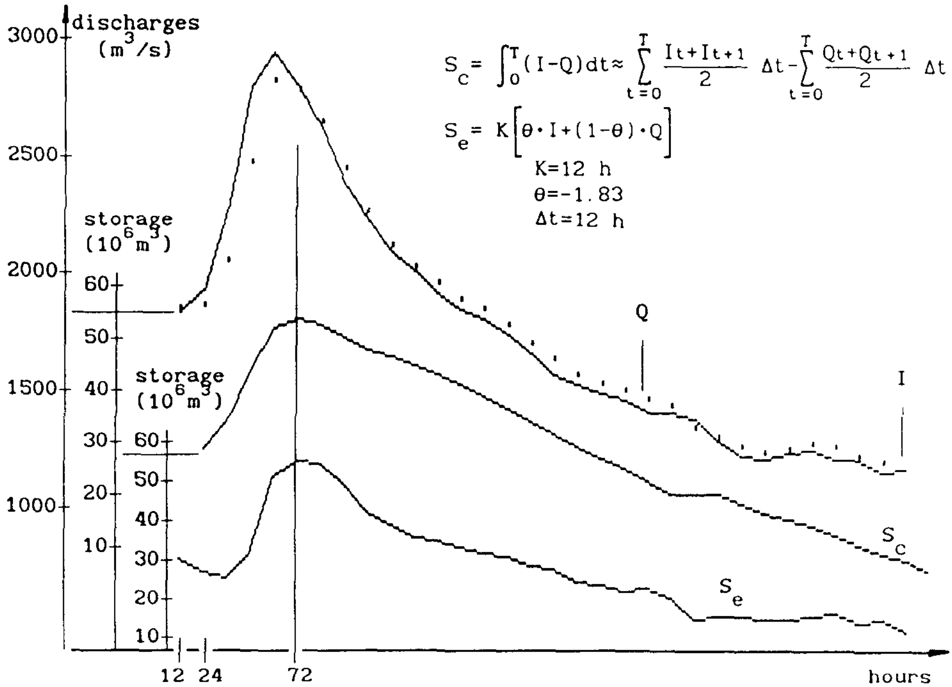

Fig. 3. Synchronous discharges of the River Danube at Budapest (upper station, continuous line) and at Dunaujvaros (lower station, dots) in discrete time steps ($\Delta t = 12$h)) as well as the calculated (S_c) and estimated storage (S_e).

simple finite difference form of the continuity eqn. (2) is used iteratively to obtain the actual 'calculated' storage S_c.

Let us consider what happens if θ with a value between zero and one is applied instead of the required negative value. In this case, the denominator of the first term in eqn. (4) is decreased. Thus the value of the term $(1/(1-\theta) \cdot (1/K(1-\theta))$ is increased, even more so because the maximum of the estimate storage is shifted back in time compared with the real storage peak so that the value of the parameter K is also decreased, although K changes to a lesser extent than θ. At the same time the term

$$\int_0^t I_r(\tau) \exp\left[-\frac{t-\tau}{K(1-\theta)}\right] d\tau \tag{6}$$

is decreased because of θ being in the exponent, so that the value of the whole first term in eqn. (4) can decrease while that of the second term of the equation increases. Thus a greater value (than the corresponding actual value) for one term is subtracted from a smaller amount (than the corresponding value) for the other term, which may result in a negative dip in outflow.

THE EFFECT OF THE NEGATIVE θ WEIGHTING PARAMETER ON THE OPTIMIZATION OF THE PARAMETERS OF THE CHANNEL ROUTING EQUATION

The continuity eqn. (2) is often written in the form of the central difference scheme (Shaw, 1983):

$$\frac{S_{t+1} - S_t}{\Delta t} = \frac{I_{t+1} + I_t}{2} - \frac{Q_{t+1} + Q_t}{2} \tag{7}$$

Substituting eqn. (1) into eqn. (7) yields (Cunge, 1969; Ponce, 1979; Perumal, 1989) the well-known channel routing equation:

$$Q_{t+1} = (C_1 \cdot I_t) + (C_2 \cdot I_{t+1}) + (C_3 \cdot Q_t) \tag{8}$$

where, for a conservative system,

$$C_1 + C_2 + C_3 = 1 \tag{9}$$

and

$$C_1 = \frac{r + 2\theta}{r + 2(1 - \theta)} \tag{10}$$

$$C_2 = \frac{r - 2\theta}{r + 2(1 - \theta)} \tag{11}$$

$$C_3 = \frac{-r + 2(1 - \theta)}{r + 2(1 - \theta)} \tag{12}$$

with

$$r = \Delta t/K \tag{13}$$

It is easy to see that the values of $C_i - s$ less than -1 or greater than $+1$ are excluded whatever negative value is assigned to the θ weighting parameter. This result is important because in practice the C_i values are optimized, and in the case when negative θ value ensures the most accurate estimation of the storage, and through this the most accurate forecasting of the discharges of the downstream section of the reach in question, the interval within which $C_i - s$ values are optimized need not be extended, and thus computation time increased.

CONCLUSIONS

The optimum value of the θ weighting parameter can be negative not only for reasons inherent in the Muskingum channel routing methods itself

(Dooge, 1973; Ponce and Theurer, 1980; Strupczewski and Kundzewicz, 1980), but also because of inadequacies of the techniques applied for measuring water stages or/and calculating corresponding discharges.

It is worth mentioning that a specific river system may have complex responses from event to event which will not be adequately described by the fixed parameters and linear representation of the Muskingum routing method, even if a negative parameter value helps in reconstituting a specific event. Other routing methods using a non-linear approach or representing more of the equations of motion are available and should be considered, when required by the circumstances of an application.

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